q-Boson Approach to Multiparticle Correlations

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An approach is proposed enabling to effectively describe, for relativistic heavy-ion collisions, the observed deviation from unity of the intercept λ (measured value corresponding to zero relative momentum $\mathbf{p}$ of two registered identical pions or kaons) of the two-particle correlation function $C(p,K)$. The approach uses q-deformed oscillators and the related picture of ideal gas of q-bosons. In effect, the intercept λ is connected with deformation parameter q. For a fixed value of q, the model predicts specific dependence of λ on pair mean momentum $\mathbf{K}$ so that, when $|\mathbf{K}| \gtrsim 500-600$ MeV/c for pions or when $|\mathbf{K}| \gtrsim 700-800$ MeV/c for kaons, the intercept λ tends to a constant which is less than unity and determined by q. If q is fixed to be the same for pions and kaons, the intercepts λ_π and λ_K essentially differ at small mean momenta $\mathbf{K}$, but tend to be equal at $\mathbf{K}$ large enough ($|\mathbf{K}| \gtrsim 800$ MeV/c), where the effect of resonance decays can be neglected. We argue that it is of basic interest to check in the experiments on heavy ion collisions: (i) the exact shape of dependence $\lambda = \lambda(\mathbf{K})$, and (ii) whether for $|\mathbf{K}| \gtrsim 800$ MeV/c the resulting λ_π and λ_K indeed coincide.

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Hadron matter under intense conditions of high temperatures and densities is extensively studied in relativistic heavy-ion collisions (RHIC). Models and approaches used to describe the processes in the reaction region are examined by comparing predictions of these models with experimental data on single-, two- and many-particle momentum spectra. Two-particle correlations are known to carry information about the space-time structure and dynamics of the emitting source [1–3]. Usually, the study of correlations occurred in RHIC assumes that (i) particles are emitted independently (completely chaotic source), and (ii) finite multiplicity corrections can be neglected. Then, correlations reflect the effects from (anti)symmetrization of the amplitude to detect identical particles with certain momenta, and the effects due to final state interactions (FSI) of detected particles between themselves and with the source. The FSI, known to depend on the structure of emitting source, as well provide certain information about source dynamics, see [4].

The correlation function is defined as [2]

$$C(\mathbf{k}_a,\mathbf{k}_b) = \frac{P_2\left(\mathbf{k}_a,\mathbf{k}_b\right)}{P_1\left(\mathbf{k}_a\right)\,P_1\left(\mathbf{k}_b\right)}\ , \tag{1}$$

$P_1\left(\mathbf{k}\right)$ and $P_2\left(\mathbf{k}_a,\mathbf{k}_b\right)$ being single- and two-particle probabilities to registrate particles with definite momenta.

For identical particles, the two-particle wave function appears to be a symmetrized (antisymmetrized) sum of single-particle states (chaoticity assumption), namely

$$\psi_{\gamma_a\gamma_b}(\mathbf{x}_a,\mathbf{x}_b,t) = \frac{1}{\sqrt{2}}\big[\psi_{\gamma_a}(\mathbf{x}_a,t)\,\psi_{\gamma_b}(\mathbf{x}_b,t)\ + \\ + e^{i\alpha}\psi_{\gamma_a}(\mathbf{x}_b,t)\,\psi_{\gamma_b}(\mathbf{x}_a,t)\big]. \tag{2}$$

Here $\alpha=0$ ($\alpha=\pi$) for identical bosons (fermions). The indices γ_a,γ_b of the 1-particle wave functions label complete sets of 1-particle quantum numbers. Below, we refer our consideration of two-particle correlations to identical bosons (pions, kaons, etc.).

In the absence of FSI, for chaotic source, the correlation function can be expressed as follows [5]:

$$C(p,K) = \\ = 1\ +\ \cos\alpha\,\frac{\left|\int d^4X\, e^{ip\cdot X} S(X,K)\right|^2}{\int d^4X\, S\Big(X,K+\frac{p}{2}\Big)\,\int d^4Y\, S\Big(Y,K-\frac{p}{2}\Big)}\ , \tag{3}$$

with the 4-momenta K and p defined as $K=\frac{1}{2}(k_a+k_b)$, $p=k_a-k_b$. The source function $S(X,K)$ (single-particle Wigner density) is defined by the emitted single-particle states $\psi_\gamma(x)$ at freeze-out times [4] (for details of the derivation see Appendix in [5]), namely

$$S(X,K) = \\ = \int d^4x\, e^{iK\cdot x}\sum_{\gamma,\gamma'}\rho_{\gamma\gamma'}\,\psi_\gamma\left(X+\tfrac{x}{2}\right)\,\psi^*_{\gamma'}\left(X-\tfrac{x}{2}\right)\ . \tag{4}$$

Using hermiticity of the density matrix $\rho_{\gamma\gamma'}$ one easily shows that $S(X,K)$ is real. Due to this, at zero relative momentum one has $C(0,K)=1+\cos\alpha$. From the latter relation it is seen that the boson correlation function at $\alpha=0$ should approach the exact value 2 as the relative momentum $\to 0$. But, as observed from the very first experiments and up to most recent data, the measured correlation function never attains this value at $\mathbf{p}=0$. To remove this discrepancy, the correlation function of identical bosons is usually represented in the form

$$C(p,K) = 1\ +\ \lambda\, f(p,K) \tag{5}$$

with λ drawn from a fit to the data, $\lambda = 0.4\ -\ 0.9$, $f(p,K)$ commonly taken as Gaussian, so that $f(\mathbf{p}=0,K)=1$. The deviation of λ from unity in RHIC can be interpreted by production of secondary pions from resonance decays going outside the fireball. Presence of long-lived resonances results in an increase of measured source size and life-times [6,7].

Trying to explain experimental data, with Eq. (3) in mind (admitting that α, besides 0 or π, can as well take other values) it is natural to relate the parameter λ with the effective angle α and to get *the reduction factor* by means of $\cos\alpha$. The correlation function (3), which is measurable quantity, possesses the obvious symmetry $\alpha\to-\alpha$, hence there is no contradiction in taking the wave function in the form (2). One can argue that the two-particle wave function of a boson pair released from a dense and hot environment effectively acquires an additional phase. The drawn phenomenon can be ascribed to properties of the medium formed in RHIC, which thus exhibits a non-standard QFT behavior through the considered correlation functions. Adopting that correlation function approaches $1+\lambda$ when the relative momentum $\mathbf{p}\to\ 0$, we construct an effective model capable to mimic real physical picture. For this, we use [5] *q-deformed* commutation relations and the techniques of *q-boson statistics* which result in a partial supression of the quantum statistical effects in many-particle systems.

As expected, usage of appropriate q-algebra allows to reduce the treatment of complex system of interacting particles to that of a system of non-interacting ones, at the price of more complicated (deformed) commutation relations. The deformation parameter q is viewed as an effective (not universal) parameter which efficiently encapsulates most essential features of complicated dynamics of the system under study. For example, in the application of q-deformed algebras to description [8] of rotational spectra of superdeformed nuclei, q is also non-universal and assumes a particular value for each nucleus. In the context of hadron theory the quantum (q-deformed) algebras also proved to be useful. Such a usage significantly improves description of hadron characteristics both in hadron scattering [9–11] (nonlinearity of Regge trajectories) and in the sector of hadron masses and mass sum rules [12].

In this letter we propose to employ, for the system of

pions or kaons produced in RHIC, the ideal q-Bose gas picture based on so-called q-bosons. Physical meaning or explanation of the origin of q-deformation in the considered phenomenon depends on whether the deformation parameter q is real or takes complex value, say, a pure phase factor. Here we do not deal with the diversity of all possible reasons (not completely unrelated) for the appearance of q-deformed statistics. Let us mention only that the *compositeness* of particles (pions, kaons) may also lead [13] to a q-deformed structure with real q.

We use the set of q-oscillators defined as [14,15]:

$$[N_i, b_j] = -\delta_{ij} b_j \ , \quad [N_i, b_j^\dagger] = \delta_{ij} b_j^\dagger \ , \quad [N_i, N_j] = 0 \ ,$$

$$[b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0 \ ,$$

$$b_i b_j^\dagger - q^{-\delta_{ij}} b_j^\dagger b_i = \delta_{ij} q^{N_j} \ . \tag{6}$$

In the Fock type representation, with the "q-bracket" defined as $[r]_q = (q^r - q^{-r})/(q - q^{-1})$, we have

$$b_i^\dagger b_i = [N_i]_q \, . \tag{7}$$

The equality $b_i^\dagger b_i = N_i$ holds only if $q = 1$ (no-deformation limit). It is required that either q is real or

$$q = \exp(i\theta) \ , \qquad 0 \le \theta \le \pi \ . \tag{8}$$

Below, just this exponential form will be adopted for q (compare with the phase α in Eqs. (2),(3)).

For a multi-pion or multi-kaon system, we consider the model of ideal gas of q-bosons (IQBG) taking the Hamiltonian in the form [16,17]

$$H = \sum_i \omega_i N_i \tag{9}$$

with i labeling energy eigenvalues, $\omega_i = \sqrt{m^2 + \mathbf{k}_i^2}$, and N_i defined as above. This is the unique truly noninteracting Hamiltonian with additive spectrum. Also, we assume discrete 3-momenta of particles, i.e. the considered system is contained in a large box of volume $\sim L^3$.

Basic statistical properties, as usual, are obtained by evaluating thermal averages $\langle A \rangle = \mathrm{Sp}(A\rho)/\mathrm{Sp}(\rho)$, $\rho = e^{-\beta H}$, now with the Hamiltonian (9). Here $\beta = 1/T$ and the Boltzmann constant is set equal to 1.

With $b_i^\dagger b_i = [N_i]_q$ and $q + q^{-1} = [2]_q = 2\cos\theta$, the q-deformed distribution function is obtained as

$$\langle b_i^\dagger b_i \rangle = \frac{e^{\beta\omega_i} - 1}{e^{2\beta\omega_i} - 2\cos(\theta) e^{\beta\omega_i} + 1} \ . \tag{10}$$

If $q \to 1$ it yields the Planck-Bose-Einstein distribution, as should, since at $q = 1$ we return to the standard system of bosonic commutation relations. Although the deformation parameter q is chosen in the *complex* form (8), the q-distribution function (10) turns out to be real, owing to dependence on q through the sum $q + q^{-1}$.

Note that the q-distribution (9) already appeared in refs. [16] (moreover, a two-parameter generalization of distribution (10) based on appropriate two-parameter deformed version of the relations (6) is also possible [18]).

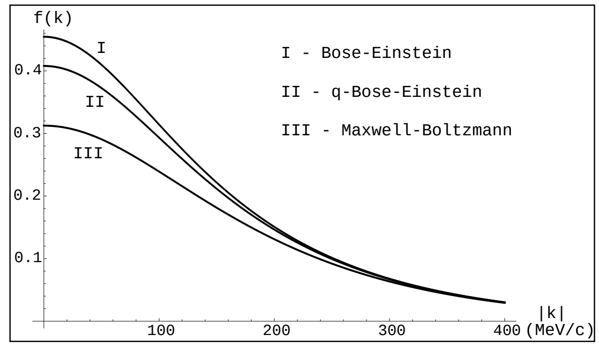

FIG. 1. The q-distribution function versus momentum (curve II), in comparison with the quantum Bose-Einstein (curve I) and Maxwell-Boltzmann (curve III) distributions. The inputs are: $T = 120$ MeV, $m = m_\pi$. Curve II corresponds to the deformation angle $\theta = 24^\circ$.

The shape of the function $f(k) \equiv \langle b^\dagger b \rangle (k)$ from (10) corresponding to the gas of pions modelled by IQBG is given in Fig. 1. As seen, the q-deformed distribution function lies completely in between the well-known two curves (standard Bose-Einstein distribution function and classical Maxwell-Boltzmann one), demonstrating that the deviation of q-distribution (10) from the quantum Bose-Einstein distribution goes in the "right direction" towards the classical Maxwell-Boltzmann one, which reflects a decreasing of the manifestation of quantum statistical effects.

Analogous curves for q-distribution functions can be given with other fixed data. For kaons, because of their larger mass and higher empirical value $\lambda \simeq 0.88$ of the intercept (which corresponds to a smaller deformation in our model), such a curve should lie closer (than pion's one) to that of the Bose-Einstein distribution.

Now let us go over to our main subject of two-particle correlations. Calculation of the two-particle distribution corresponding to the q-oscillators (6) yields

$$\langle b_i^\dagger b_i^\dagger b_i b_i \rangle = \frac{2\cos\theta}{e^{2\beta\omega_i} - 2\cos(2\theta) e^{\beta\omega_i} + 1} \ . \tag{11}$$

Then, the desired formula for the intercept $\tilde{\lambda}_i \equiv \lambda_i + 1 = \langle b_i^\dagger b_i^\dagger b_i b_i \rangle / (\langle b_i^\dagger b_i \rangle)^2$ of two-particle correlations follows,

$$\lambda_i + 1 = \frac{2\cos\theta (t_i + 1 - \cos\theta)^2}{t_i^2 + 2(1 - \cos^2\theta) t_i} \tag{12}$$

with $t_i = \cosh(\beta\omega_i) - 1$. Note that (11), (12) are real functions since, like (10), these depend on the q-parameter (8) through the combination $\frac{1}{2}(q + q^{-1}) =$

$\cos\theta$. Below, we drop the subscript i and ignore the above assumed discreteness of momenta.

The quantity λ can be directly confronted with empirical data. In the limit $q \to 1$, from Eq. (12) the intercept $\lambda_{\rm BE} = 1$ proper for Bose-Einstein statistics is reproduced. This corresponds to Bose-Einstein distribution, see (10) at $\theta = 0$. On the other hand, at $\theta = \pi/2$, Eq. (12) coincides with the value $\lambda_{\rm FD} = -1$ proper for Fermi-Dirac statistics. The two cases, $\lambda_{\rm BE}$ and $\lambda_{\rm FD}$, are seen in Fig. 2 as the only two points where all the different curves (the continuum parametrized by $w = \beta\omega$) merge and the dependence on momentum and temperature disappears. From the continuum of curves, there exists a unique limiting asymptotic one $\tilde\lambda = 2\cos\theta$ (or $\lambda = -1 + 2\cos\theta$), which corresponds to $w \to \infty$, i.e. to low temperature or to large momentum.

Let us discuss implications of Eq. (12). Solving it for $\cos\theta$ at a fixed $\lambda = \lambda_1$, we obtain the deformation angle as the function $\theta = \theta(\lambda_1,\ \mathbf{K},\ T,\ m)$. Fig. 2 illustrates the properties of intercept (correlation strength) λ treated from the standpoint of q-deformation, i.e., on the base of Eq. (12). Note that the continuum of curves $\tilde\lambda = \tilde\lambda(\cos\theta)$ parametrized by $w = \beta\omega$ divides into three classes ("subcontinua") given by the intervals: (i) $0 < w \le w_0$, (ii) $w_0 < w < w_0'$, and (iii) $w_0' \le w < \infty$. The two "critical" values $w_0 = w_B \simeq 0.481$ and $w_0' = w_D \simeq 0.696$ fix the curves B and D respectively. The curves A, C, E are typical representatives of the classes (i), (ii), (iii). All the curves from classes (i), (ii) possess two extrema; the curve D is unique since its extrema degenerate, coinciding with the point of inflection. This enables us to define the range of "small deformations" $I_{\rm small}$ for the variable θ: from $\theta = 0$ (no deformation) to the value yielding minimal λ, $\lambda_{\rm min} \approx 0.33$, implied by the "critical" value $w_0' = w_D$. On the interval $I_{\rm small}$ the intercept λ monotonically decreases with increase of θ (or $1 - \cos\theta$, the strength of deformation).

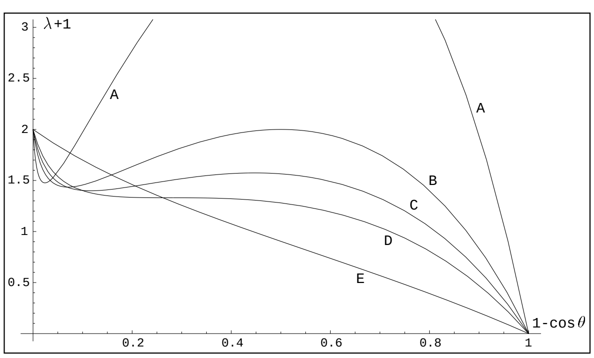

FIG. 2. The $(1 - \cos\theta)$-dependence of the intercept λ of two-particle correlation, given by Eq. (12). The curves A,B,C,D,E correspond to the values $w_A = 0.3$, $w_B = 0.481$, $w_C = 0.58$, $w_D = 0.696$, $w_E = 2.0$ of the dimensionless variable $w \equiv \beta\omega$.

All the type (ii)–(iii) curves at $\theta \ne 0$ lie below the line $\tilde\lambda = 2$ — largest possible correlation attainable in Bose-Einstein case (the curve B contains, besides $\theta = 0$, *single* point at certain value of $1 - \cos\theta$, where $\tilde\lambda = 2$ is also attained). For the type (ii) curves we restrict ourselves with $I_{\rm small}$, ignoring moderately (and very) large deformations. The class (i) contains "irregular" curves for each of which there exist q-deformations yielding correlation strengths exceeding $\tilde\lambda = 2$. Consider special values of physical variables T, $|\mathbf{K}|$, which provide the peculiar values $w_0 = w_B \simeq 0.481$ and $w_0' = w_D \simeq 0.696$ (recall that $w = \sqrt{m^2 + \mathbf{K}^2}/T$). With $m(\pi^{\pm,0}) = 139.57$ MeV and lowest mean momentum of the pion pair $|\mathbf{K}| = 0$, we get the respective two lower bounds for the temperature: $T_0 = 290.0$ MeV and $T_0' = 200.5$ MeV. Compare these values with that for a typical curve from class (iii): at $w = w_E = 2.0$ (the curve E) for pions with $|\mathbf{K}| = 0$ we get the lowest temperature $T_E = 69.8$ MeV.

The results similar to presented above (starting with Eq. (6)) were also given [5] for q-oscillators with defining relation $aa^\dagger - qa^\dagger a = 1$ introduced by Arik and Coon [9].

Theoretical approaches to RHIC are aimed to find an adequate description for the non-equilibrium state formed during the collision, and the q-boson technique enables us to treat the non-stationary hot and dense matter effectively as a "noninteracting ideal gas". To determine the q-parameter corresponding to actual state of the hot medium, we propose a way of extracting a useful information from the two-particle correlations. In a more general context, by this we propose and develop an effective picture for the two-particle correlations in RHIC.

Let us specially emphasize a remarkable feature exhibited within our model:

At $w \to \infty$ (i.e. for very low temperature at fixed momenta or very large momenta at fixed temperature), see Fig. 2 and relevant comments, we come to the equality

$$\lambda = 2\cos\theta - 1 \qquad (T \to 0 \ \text{ or } \ |\mathbf{K}| \to \infty) \tag{13}$$

for the employed q-oscillators (6). This means a direct link $\lambda \leftrightarrow q$ whose explicit form is $\tilde\lambda = [2]_q = 2\cos\theta$.

Finite temperature and momenta, on the other hand, become non-trivially involved in the relation between λ and the parameter q as seen from Eq. (12) and Fig. 2.

In Fig. 3, we present the dependence, implied by Eq. (12), of intercept λ on the momentum $|\mathbf{K}|$ exemplifying this for pions at $T = 120$ MeV (and at $T = 180$ MeV) with four curves corresponding to the fixed values 4°, 9°, 15°, 24° of the deformation angle θ. Analogous dependence for the case of kaons is presented in Fig. 4. Each curve has its own asymptote given by Eq. (13) and lying beneath the line $\lambda = 1$. Recall that, if the unique cause for reducing of the intercept would be decays of resonances (commonly accepted idea), all the curves would tend to the value $\lambda = 1$ in their asymptotics. In this respect, on the contrary, our model clearly suggests possible existence of some other reasons, than just the decays of resonances, for the deviation of λ from

unity. This property probably reflects complicated dynamics of strongly interacting system prior to freeze-out. The particular kind of the behaviour of λ with respect to $|\mathbf{K}|$ (implied for a fixed deformation), is a direct consequence of our model.

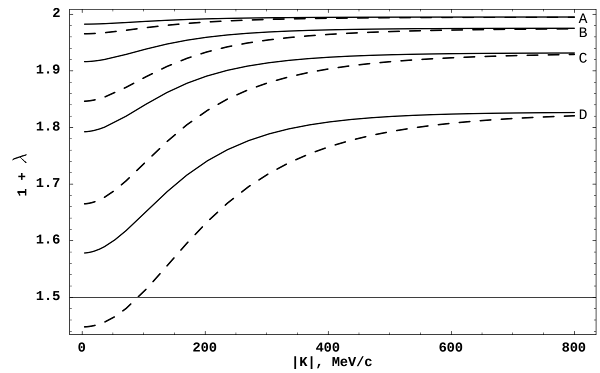

FIG. 3. The intercept λ versus pion momentum $|\mathbf{K}|$ for fixed θ: A) $\theta = 4°$, B) $\theta = 9°$, C) $\theta = 15°$, D) $\theta = 24°$. The inputs are $T = 120$ MeV - solid curve, $T = 180$ MeV - dashed curve, and $m = m_\pi$.

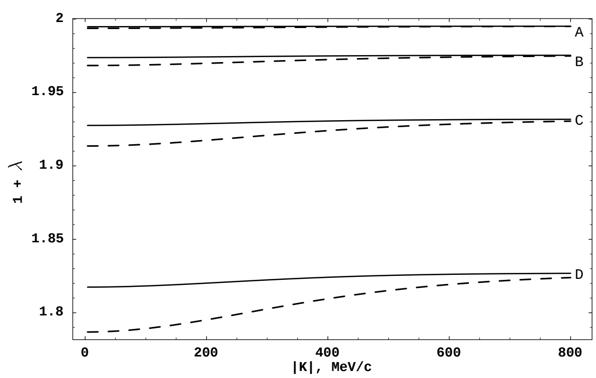

FIG. 4. The intercept λ versus kaon momentum $|\mathbf{K}|$ for fixed θ, with the same as in Fig. 3 values of the parameters θ, T, and now $m = m_K$.

Let us summarize main points of the present letter. In contrast to the commonly accepted opinion that the intercept λ of the two-pion correlations should asymptotically approach the value one, we predict attaining by λ the value less than one at large pion (kaon) pair mean momentum $|\mathbf{K}|$. This effect can be verified in the current experiments which take place on the SPS at CERN and on the Relativistic Heavy Ion Collider at BNL. Therefore we propose to pay special attention, in these experiments, to the behaviour of the intercept λ at large enough pion (kaon) pair mean momentum, that is, $|\mathbf{K}|$ in the range up to $500-600$ MeV/c for pions (up to $700-800$ MeV/c for kaons) in the fireball frame. If our prediction is confirmed, this means that λ encapsulates certain memory effects: it carries important information about the fireball dynamics, especially concerning probable quark-gluon plasma phase, hadronization, other nonstationary processes which took place in the fireball before freeze-out.

As follows from our model, at large enough pair mean momenta of the two pions or two kaons, the dependence of intercept λ on temperature as well as on the particle mass (and energy) is washed out. Due to this, λ_π and λ_K should practically coincide already at $|\mathbf{K}| \simeq 800$ MeV/c, although at much smaller mean momenta the intercepts λ_π and λ_K significantly differ from one another (compare the corresponding curves in Fig. 3 and Fig. 4, labelled by the same θ). Let us recall that this remarkable feature is the result of asymptotical reducing of the full T-, $\mathbf{K}$-, and m-dependence contained in Eq. (12) to the direct link between the intercept λ and the deformation parameter q (with $q = \exp \mathrm{i}\theta$), given by Eq. (13). Therefore, *we put forward two principal points for experimental verification*:
1) The shape of dependence of intercept λ on the pair mean momentum as well as its asymptotics (constant value of λ_π, λ_K less than unity at large $|\mathbf{K}|$);
2) Coincidence of the intercepts for different pairs of particles at large enough mean momentum, for instance, the regime with $\lambda_\pi \approx \lambda_K$ should manifest itself starting already from $|\mathbf{K}| \simeq 800$ MeV/c.

Verification of these issues is of fundamental interest. Indeed, the first point will give answer to the question: is the q-deformed description (q-boson field theory) really capable to mimic the unusual properties of "hot" and "dense" hadron matter? In particular, can we adopt this description as an efficient tool to deal with nonequilibrium, short-lived system of pions or kaons at high temperature and density? The second point of verification will clarify whether one can regard the deformation parameter q as a kind of quantity, which may be referred to as one of basic characteristics of the hot, short-lived system (besides "temperature" and chemical potential).

We hope that the answers to these intriguing questions will be deduced in the nearest future in the experiments which take place in CERN and BNL.

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